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International Journal of Solids and Structures 39 (2002) 5571–5583

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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# Multiwave non-linear couplings in elastic structures Part 1. One-dimensional examples

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Received 30 October 2001; received in revised form 20 February 2002

**Abstract**

This short contribution has for main purpose to exhibit the essentials of non-linear wave properties in mechanical systems of engineering origin (structural members). Non-linear resonance is examined on two one-dimensional examples, an infinite straight elastic bar and a thin elastic circular ring, exhibiting continuous and discrete spectra, respectively. Three-wave and four-wave interactions and the stability of coupled modes with respect perturbations are discussed, the emphasis being placed on mechanical phenomena (e.g., stress amplification), although analogies with some non-linear optical systems are obvious.


## 1. Introduction

The phenomenon of *non-linear resonance coupling* classically occurs in physical systems that are governed by distinct modes of propagation; these may be of the same physical nature—e.g., various mechanical modes-, or they may be of totally differing natures—say, mechanical and magnetic or electric—as is often the case. In any case the two basic ingredients needed are (i) the existence of *multimodes* in the physical system and (ii) the *dispersion* of these modes in the linearized case. Such physical situations have received the attention of applied mathematicians and wave specialists in various fields of physics and engineering science, e.g., in *non-linear optics and radiophysics* (cf. Sukhorukov, 1988; Nelson, 1979), in *fluid dynamics* (cf. Craig, 1985), and in *elastic crystals with a microstructure* (cf. Maugin, 1999; Potapov et al., 1998). In the case of elastic crystals the multimodes are due to a coupling of classical elastic degrees of freedom with the kinematics of an internal structure—a rigid mechanical one such as in micropolar media and liquid crystals, a magnetic one such as in ferromagnets (coupling between phonons and magnons), and an electric one in ferroelectric bodies (electroelastic couplings).

In the present work, we focus attention on the non-linear wave couplings in *engineering elastic structures*. More particularly in this part on *one-dimensional examples*, one such structure is an elastic infinitely long

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PII: S0020-7683(02)00365-7

straight bar and the other is a thin closed circular ring. These two structures are chosen because they exhibit a continuous spectrum and a discrete one (due to the circular periodicity), respectively. They have, therefore, the value of paradigms. They are perfect examples of non-linear oscillatory systems exhibiting a hierarchy of wave instabilities. The mathematical tools used are those of non-linear science, essentially *asymptotics*. The related algebra often is cumbersome and will, therefore, be omitted most of the time. It can be found in lengthy original reports. We emphasize here the mechanical consequences of the analysis.

## 2. Non-linear waves in a thin infinitely long bar

In a non-dimensional notation the relevant basic field equations are the following ones (Kauderer, 1958):

$$\begin{aligned} u_{tt} - u_{xx} &= \frac{\mu}{2}\partial_x w_x^2, \\ w_{tt} + \alpha^2 w_{xxxx} &= \mu\partial_x(u_x w_x) + \frac{\mu^2}{2}\partial_x w_x^3, \end{aligned} \tag{2.1}$$

where $u$ is the longitudinal displacement of the middle line of the bar, $w$ is the transverse displacement, $\alpha$ is the non-dimensional radius of inertia of the bar, and $\mu$ is a coupling parameter supposed to be sufficiently small to justify the validity of asymptotic considerations. Eq. (2.1) are established under the working hypotheses of Bernoulli and Euler. Only second-order couplings between the *longitudinal mode* $u$ and the *bending mode* $w$ are kept at most. The linear analysis of Eq. (2.1) yields straightforwardly the modes as non-dispersive direct and counter propagating longitudinal waves of frequency

$$\omega_l = \pm k \tag{2.2}$$

and highly dispersive bending waves of frequency

$$\omega_b = \pm\alpha k^2. \tag{2.3}$$

The spectra are sketched in Fig. 1.

Now we consider the possible coupling between *three* waves selected at working points 1, 2 and 3 in this figure in a typical parallelogram form such that we satisfy the so-called *three-wave phase matching*

$$\omega_3 = \omega_1 + \omega_2, \qquad k_3 = k_1 \pm k_2, \quad \omega_3 \geqslant \omega_2 \geqslant \omega_1. \tag{2.4}$$

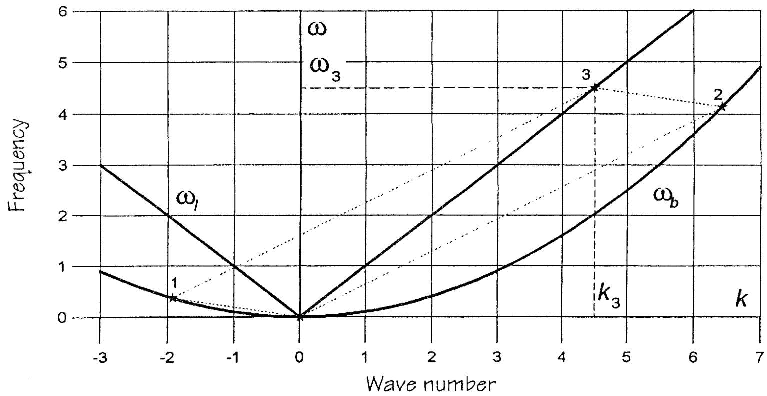


Fig. 1. Spectrum and three-wave resonance for a thin infinitely long bar.

That is, we consider the problem of *modal energy exchange* between a large-amplitude high-frequency longitudinal wave (point 3) coupled to two low-frequency bending waves (points 2 and 1) propagating in opposite directions. The three frequencies thus selected are said to form a *resonant triad*, or in a more music-like Pithagorean fashion, a *resonant trio*. The non-linear resonant coupling between these modes is now examined on the basis of the Eq. (2.1) at order one in the small coupling parameter $\mu$. Coupled solutions are sought in the form

$$\begin{aligned} u(x,t) &= A_3(\chi,\tau)\exp i\Phi_3 + \mu u^{(1)}(x,t) + (\cdot\cdot)^*, \\ w(x,t) &= A_1(\chi,\tau)\exp i\Phi_1 + A_2(\chi,\tau)\exp i\Phi_2 + \mu w^{(1)}(x,t) + (\cdot\cdot)^*, \end{aligned} \tag{2.5}$$

where $\chi = \mu x$, $\tau = \mu t$, $\mu \ll 1$, the $A_n$ are slowly varying amplitudes of the parts of (2.5) that are solutions satisfying the linear field equations (they are thus determined by initial and boundary conditions), and the phases $\Phi_n$ are such that

$$\Phi_n = \omega_n t - k_n x, \tag{2.6}$$

with each couple $(\omega_n, k_n)$ satisfying the correspondingly numbered dispersion relation and altogether the phase matching conditions. The symbolism $(\cdot\cdot)^*$ denotes complex conjugacy. On substituting from (2.5) into Eq. (2.1) and averaging the resulting equations over the phases $\Phi_n$, we obtain a system of three coupled hyperbolic partial differential equations for the amplitudes $A_n$:

$$\frac{\partial A_n}{\partial \tau} + v_n \frac{\partial A_n}{\partial \chi} = \frac{\beta}{\omega_n} \frac{\partial U}{\partial A_n^*}, \tag{2.7}$$

where

$$v_n = \mathrm{d}\omega_n(k_n)/\mathrm{d}k, \tag{2.8}$$

$$\beta = -k_1 k_2 k_3 / 2 \tag{2.9}$$

and

$$U = A_1 A_2 A_3^* + A_1^* A_2^* A_3 \tag{2.10}$$

are, respectively, the group velocities of the "linear" modes, a coefficient of *non-linearity*, and what may be called a *cubic average potential*. The Cauchy problem associated with (2.7) requires the knowledge of the initial conditions $A_n(\chi, 0) = a_n(\chi)$, $n = 1, 2, 3$.

On setting

$$E_n = \omega_n^2 |A_n|^2, \qquad S_n = v_n E_n \tag{2.11}$$

the energy and energy flux associated with each linear mode, we can establish several consequences of Eqs. (2.7) and (2.8) such as the equation

$$\frac{\partial}{\partial \tau}(E_1 + E_2 + E_3) + \frac{\partial}{\partial \chi}(S_1 + S_2 + S_3) = 0, \tag{2.12}$$

clearly a law of *conservation of energy* between the three modes, and equations of the type

$$\frac{\partial}{\partial \tau}\left(\frac{E_1}{\omega_1} - \frac{E_2}{\omega_2}\right) + \frac{\partial}{\partial \chi}\left(\frac{S_1}{\omega_1} - \frac{S_2}{\omega_2}\right) = 0 \tag{2.13}$$

and similar ones by permutation. Eqs. (2.12) and (2.13) are *canonical* (as such, they are formally identical to those obtained for three-wave mixing in non-linear optics; cf. Nelson, 1979). Direct consequences of these are the well known *Manley-Rowe relations* (first integrals of Eq. (2.12) and (2.13) that characterize the energy partition between modes), see Kovriguine and Potapov (1996):

$$E = E_1 + E_2 + E_3 = \text{const.}, \tag{2.14}$$

$$\frac{E_1}{\omega_1} - \frac{E_2}{\omega_2} = C_1, \quad \frac{E_2}{\omega_2} + \frac{E_3}{\omega_3} = C_2, \quad \frac{E_1}{\omega_1} + \frac{E_3}{\omega_3} = C_3, \tag{2.15}$$

where the $C_n$ are constants.

**Remark.** For spatially uniform processes ($\partial/\partial\chi \to 0$), Eq. (2.7) yield the reduced equations

$$\frac{dA_1}{d\tau} = \frac{\beta}{\omega_1} A_3 A_2^*, \quad \frac{dA_2}{d\tau} = \frac{\beta}{\omega_2} A_1^* A_3, \quad \frac{dA_3}{d\tau} = \frac{\beta}{\omega_3} A_1 A_2. \tag{2.16}$$

These are identical to the Euler equations of motion for a rigid body about a fixed point (for real-valued variables, obviously; (cf. Landau and Lifshitz, 1976)).

At the degree of approximation (cf. Eq. (2.5)) of the present approach, we have the following easily established results concerning the stability of modes:

(i) Longitudinal waves are *unstable* with respect to small low-frequency perturbations (so-called *break-up instability*),
(ii) Bending waves are *stable*—(at least) within the present first-order non-linear approximation—with respect to small high-frequency perturbations.
(iii) The loss of stability against the high-frequency wave can lead to a *dynamic stress growth* caused by the resonant excitation of two low-frequency waves. As a consequence, one must pay special attention to the initial stress level, e.g., one may envisage a restriction on it so as to stay in the elastic regime.

Finally, one may inquire about the *temporal evolution* of the considered triad. This requires exploiting a technique such as the *inverse scattering method* (ISM; cf. Kovriguine et al., 2001) to find out analytical expressions (in terms of Jacobian elliptic functions). Fig. 2 gives such an evolution for amplitudes of relatively close sizes. One may also remark that the physical system considered may exhibit *triple-wave envelope solitons*—the three amplitudes travel then together as a "complex of solitonic shapes"—in which case (illustrated in Fig. 3) modes 1 and 2 are "bright" solitons and mode 3 is a "dark" one (in the optical

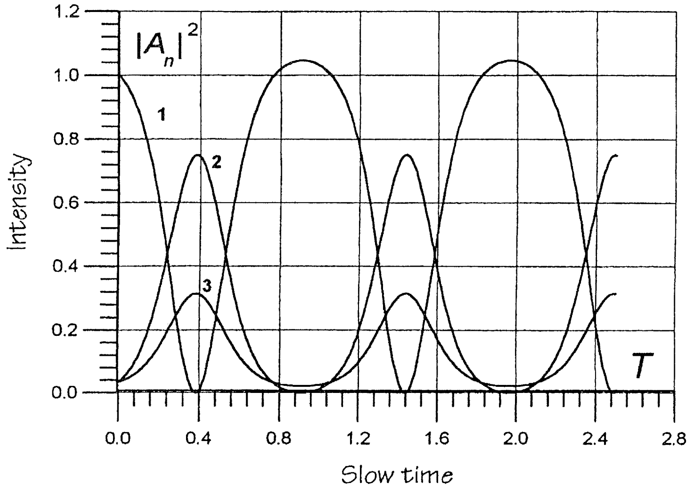


Fig. 2. Evolution of squared amplitudes for the triad of Fig. 1.

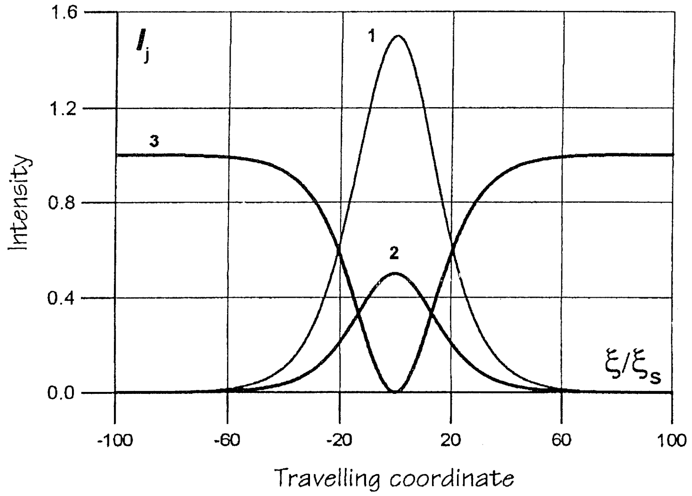


Fig. 3. Soliton-like solution without energy exchange for the triad of Fig. 1.

jargon) so that energy is conserved. In this case, the amplitudes being fixed once and for all, there is *no* energy exchange while the triad travels inertially at constant speed (cf. Fig. 3).

### 2.1. The problem of four-wave resonant interactions

We now briefly investigate the influence of higher-order couplings—those of order $\mu^2$—on the stable (bending) waves. This also allows one to consider four-wave interactions—something much less studied than three-wave ones (cf. Craig, 1985, in fluid mechanics), to confirm the generation of non-linear stationary waves due to *self-modulation*, and to examine the evolution for finite-amplitude bending waves.

Extending somewhat the ansatz (2.5), we now seek solutions in the form

$$\begin{aligned} u(x,t) &= A(\chi,\tau) + \mu u^{(1)}(x,t) + \mu^2 u^{(2)}(x,t) + \mathrm{O}(\mu^3 t) + (\cdot\cdot)^*, \\ w(x,t) &= \sum_{n=1}^{4} B_n(\chi,\tau)\exp \mathrm{i}\Phi_n + \mu w^{(1)}(x,t) + \mu^2 w^{(2)}(x,t) + \mathrm{O}(\mu^3 t) + (\cdot\cdot)^*, \end{aligned} \tag{2.17}$$

where $A(\chi,t)$ is a nearly uniform non-oscillatory perturbation, $B_n(\chi,t)$ are slowly varying amplitudes, the phases $\Phi_n$ are such as in (2.6) with $\omega_n$ and $k_n$ satisfying the linear dispersion relations, and $u^{(\mathrm{m})}$ and $w^{(\mathrm{m})}$ are *finite resonant* corrections.

On substituting from (2.17) in (2.1) at the first order in $\mu$, we obtain the following system for $u^{(1)}$ and the $B_n$'s:

$$(\partial_\tau + v_n\partial_\chi)B_n(\chi,\tau) = 0, \tag{2.18}$$

$$u_{tt}^{(1)} - u_{xx}^{(1)} = \sum_{n,m=1}^{4} \frac{\mathrm{i}k_n k_m}{2}[k_{n-m}B_n B_m^* \exp \mathrm{i}\Phi_{n-m} - k_{n+m}B_n B_m \exp \mathrm{i}\Phi_{n+m}] + (\cdot\cdot)^*, \tag{2.19}$$

where we have set

$$v_n = \mathrm{d}\omega_n/\mathrm{d}k, \quad k_{n\pm m} = k_n \pm k_m, \quad \Phi_{n\pm m} = \Phi_n \pm \Phi_m. \tag{2.20}$$

Eq. (2.18) represents a first-order wave equation for each $B_n$. Eq. (2.19) is far too complicated in this general state. It however shows that the longitudinal component $u^{(1)}$ is driven by quasi-harmonic bending waves so that $u_x^{(1)}$ will involve combinational harmonics. In order to provide a more specific illustration, consider the case where the $\omega_n$ and $k_n$ satisfy the following *phase-matching* conditions:

$$\omega_1 = \omega_2 + \omega_3 + \omega_4 + \mu^2 \Delta\omega, \qquad k_1 = k_2 \pm k_3 \pm k_4, \tag{2.21}$$

where the so-called *frequency detuning* $\Delta\omega$ is of the order of $\max \omega_n$. Such a situation is illustrated in Fig. 4. We say that the four working points on this dispersion diagram provide a *resonant quartet* (there are other possibilities providing other *quartets* (cf. Kovriguine et al., 2001)). It is then of interest to look at the next order of approximation ($\mu^2$) for the solution of the system (2.1). This now provides a system of equations that govern the $B_n$'s and the amplitude $A$ of the longitudinal mode. This system reads thus

$$\begin{aligned} &\left(\partial_\tau + v_n \partial_\chi + \frac{\mathrm{i}\mu v_n'}{2}\right) B_n = \frac{\mathrm{i}\mu}{2\omega_n}\left[\gamma_0 \frac{\partial U}{\partial B_n^*} + \sum_{n=1}^{4} B_n(\gamma_{nm}|B_m|^2 + k_n^2 A_\chi)\right], \\ &A_{\tau\tau} - A_{\chi\chi} = \sum_{n=1}^{4} k_n^2 \partial_\chi |B_n|^2, \end{aligned} \tag{2.22}$$

where $\gamma_0$ is a non-linearity coefficient, the coefficients $\gamma_{nm}$ depend on $k_n$, $k_m$, $v_n$ and $v_{n\pm m}$ (see their definitions in Kovriguine et al., 2001), $v_n' = \mathrm{d}^2\omega_n/\mathrm{d}k^2$ is the curvature of the spectrum of mode $n$ and

$$U = B_1 B_2^* B_3^* B_4^* \exp\mathrm{i}(\Delta\omega)\tau + B_1^* B_2 B_3 B_4 \exp\mathrm{i}(-\Delta\omega)\tau \tag{2.23}$$

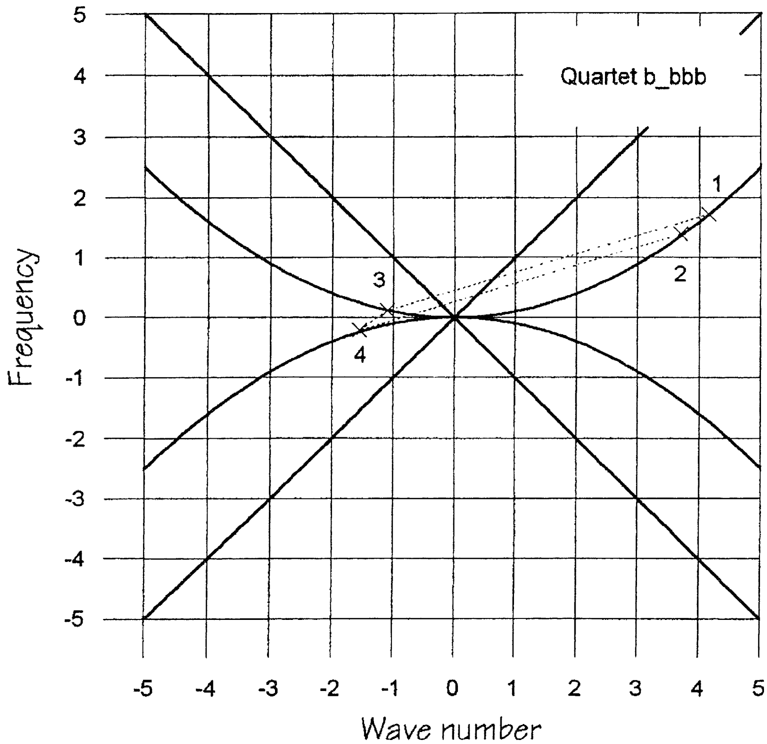


Fig. 4. Four-wave resonance in the thin infinitely long bar.

is the so-called four-wave resonant interaction *potential*. Eqs. (2.22) and (2.23) possess a Hamiltonian structure with first (energy) integral $H = H(A, B_n; n = 1, 2, 3, 4)$. This allows one, knowing analytical solutions, to estimate the stability of these solutions.

Note that as $\tau \to \infty$, we asymptotically obtain

$$A_\chi = \sum_{n=1}^{4} \frac{k_n^2 \partial_\chi |B_n|^2}{v_n^2 - 1}. \tag{2.24}$$

This, by $\chi$ integration, can be interpreted as the fact that low-frequency bending waves (such that $v_n < 1$) induce *uniform stretching* proportional to the wave intensity $|B_n|^2$. This is some kind of *non-local effect* akin to a mean flow in hydrodynamics.

Typical temporal patterns (squared-amplitude variations versus slow time $\tau$) for the above-selected quartet have been computed and are illustrated in Fig. 5. Further remarks about this case concern three points:

(i) The problem examined is quite similar to one met in other areas of non-linear science. In particular, a similar analysis prevails in (a) the study of quasi-one-dimensional wave packets on the upper ocean, (b) the study of interactions between capillarity and gravity waves in a fluid, (c) that of the interaction of Langmuir and ion-acoustic waves in plasmas, (d) that of exciton–phonon interactions in molecules of proteins, (e) in the study of the evolution of packets of surface waves, and (f) that of electron–phonon interactions in crystals.

(ii) Like in the other physical problems mentioned at point (i) there also exist non-resonant interactions, when some of the four waves have zero amplitude. In particular, the study of cross interactions (or cross modulations) then yields a system of *non-linearly coupled Schrödinger equations*, and the self-modulation of bending-wave trains and envelope solitons, yields a *generalized Zakharov system* in the form introduced by Maugin et al. (1992) in the study of elastic surface waves on stratified structures—see these in Kovriguine and Potapov (1998), and in the long report by Kovriguine et al. (2001).

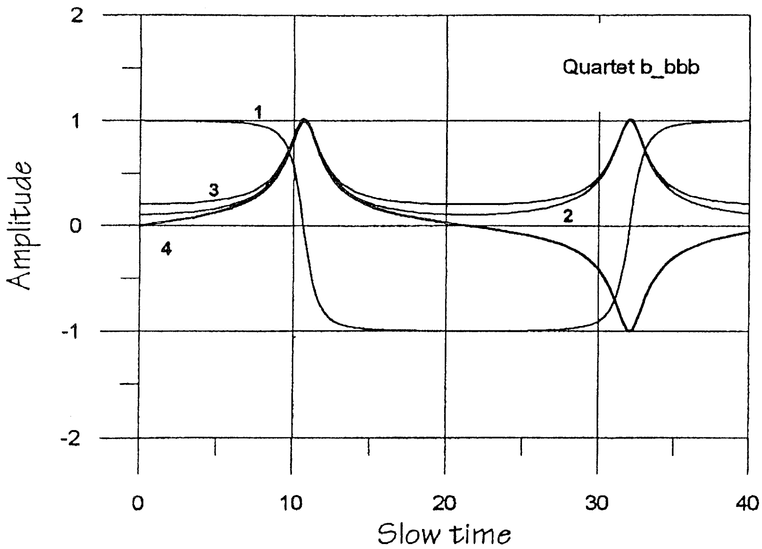


Fig. 5. Evolution of squared amplitudes for the quartet of Fig. 4.

## 3. Waves traveling around a closed circular ring

This case is only briefly sketched out in order to emphasize the differences with the infinite bar of Section 2. In this new case which necessarily implies *periodicity*, the ring is viewed as a slice of an infinitely long thin shell and it obviously presents a discrete spectrum circumferentially. Special attention is paid to the dynamical loss of stability against axisymmetric oscillations caused by a radially uniform impact. The non-linearity yields non-trivial dynamical effects and we observe the existence, via modal exchanges of energy, of *resonant triads* between high-frequency axisymmetric oscillations and bending travelling waves (of equal wave numbers), leading eventually to the instability of low-frequency bending waves (at the second order of approximation). The basic equations are those of thin walled shells in the geometrically non-linear theory. Let $\varphi$ the azimuthal angle and $R$ the radius of the ring of thickness $h$. If $v(\varphi,t)$ and $w(\varphi,t)$ are the non-dimensional azimuthal displacement of the middle line and the non-dimensional transverse (radial) displacement, it is convenient to introduce the following change of dependent variables:

$$V = v_\varphi + w, \qquad W = w_\varphi - v. \tag{3.1}$$

Accordingly, the equations governing $v$ and $w$ are derived from a Lagrangian formulation with Lagrangian $L$ such that

$$L = K - \Pi, \tag{3.2}$$

with kinetic and potential energies given (in dimensional units) by

$$K = \frac{1}{2}\rho h(v_t^2 + w_t^2), \quad \Pi = \frac{1}{2}\int_{-h/2}^{+h/2} E\varepsilon_{\varphi\varphi}^2\,\mathrm{d}\zeta, \tag{3.3}$$

where $E$ is Young's modulus, and the azimuthal strain is given by (Ginsberg, 1974)

$$\varepsilon_{\varphi\varphi} = \frac{1}{R}(v_\varphi + w) - \frac{\zeta}{R^2}(w_\varphi - v)_\varphi + \frac{1}{2R^2}(w_\varphi - v)^2, \tag{3.4}$$

where $\zeta \in (-(1/2)h, +(1/2)h)$ is the radial distance from the middle line in a cross-section of the ring. The corresponding non-dimensional field equations, on account of (3.1) are deduced as

$$\begin{aligned} v_{tt} - V_\varphi + \varepsilon^2 W_{\varphi\varphi} &= \mu\left[\frac{1}{2}(W^2)_\varphi + WV\right] + \frac{\mu^2}{2}W^3, \\ w_{tt} + V + \varepsilon^2 W_{\varphi\varphi\varphi} &= \mu\left[(VW)_\varphi - \frac{1}{2}W^2\right] + \frac{\mu^2}{2}(W^3)_\varphi. \end{aligned} \tag{3.5}$$

Here again $\mu \ll 1$, and the relative thickness of the ring is defined by $\varepsilon = h/\sqrt{12}R$. The latter eventually provides a second small parameter.

The circular shape of the ring imposes that solutions of (3.5) satisfy the periodicity conditions

$$v(\varphi,t) = v(\varphi + 2\pi, t), \qquad w(\varphi,t) = w(\varphi + 2\pi, t). \tag{3.6}$$

If $\tau = \mu t$ is the slow time, the linear-wave phases will be of the form

$$\Phi_{k,n} = \omega_{k,n}\tau + n\varphi, \tag{3.7}$$

where $n$ is an integer. To each one of the angular frequencies $\omega_{k,n}$ there will correspond a *normal wave*. We can play with the existence of the second small parameter $\varepsilon$. In particular we may consider the case of rings with small curvature (large radius) for which $\varepsilon \ll 1$. Then the *linear* coupling between modes in Eq. (3.2) with vanishing right-hand sides can be said to be *weak*. On discarding $\mu$-terms, Eq. (3.5) yield the following separate dispersion relations whenever we implement the *inextensibility condition* of the middle line of the ring (cf. Donnell, 1979): $V = 0$, i.e., $w_\varphi = v$ (note this is *not* the same condition as $\varepsilon = 0$):

$$\omega_{1,n} = \pm\varepsilon n \frac{n^2 - 1}{\sqrt{n^2 + 1}}, \qquad \omega_{2,n} = \pm\sqrt{n^2 + 1} \tag{3.8}$$

for bending and azimuthal waves, respectively. In these conditions it is shown that the amplitudes of the azimuthal, $A_{k,n}$, and bending, $B_{k,n}$, components of the normal waves are linearly related by

$$B_{k,n} = -\mathrm{i}p_{k,n}A_{k,n}, \tag{3.9}$$

with coefficients $p_{k,n}$, $k = 1, 2$, given approximately by

$$p_{1,n} \approx n^{-1}, \qquad p_{2,n} \approx -n. \tag{3.10}$$

The linear dynamical solution (3.8)–(3.10) is usually considered as quite satisfactory. In order to respect the long-wave limit approximation, the values of $n$ should be bounded from above by some maximal wave number $n_{\max}$, e.g., by a characteristic wavelength that should not exceed a certain number of ring thicknesses (e.g., $\lambda \geqslant 10h$). If the simplifying hypotheses applied to obtain the approximations (3.8) do not apply, then we cannot discard the terms in $W$ in the left-hand sides of Eq. (3.5). The linear dispersion relations then read

$$\omega_{k,n}^2 = \tfrac{1}{2}(n^2 + 1)(1 + \varepsilon^2 n^2) \mp \tfrac{1}{2}\left((n^2 + 1)^2(1 + \varepsilon^2 n^2)^2 - 4n^2\varepsilon^2(n^2 - 1)^2\right)^{1/2}. \tag{3.11}$$

The proportionality coefficients $p_{k,n}$ of the amplitudes in Eq. (3.9) are now given by the more complicated frequency-dependent formula

$$p_{k,n} = \frac{n(1 + \varepsilon^2 n^2)}{(1 + \varepsilon^2)n^2 - \omega_{k,n}^2}, \tag{3.12}$$

with the orthogonality condition $p_{1,n}p_{2,n} = -1$. The spectra for a chosen value of $\varepsilon$ are shown in Fig. 6.

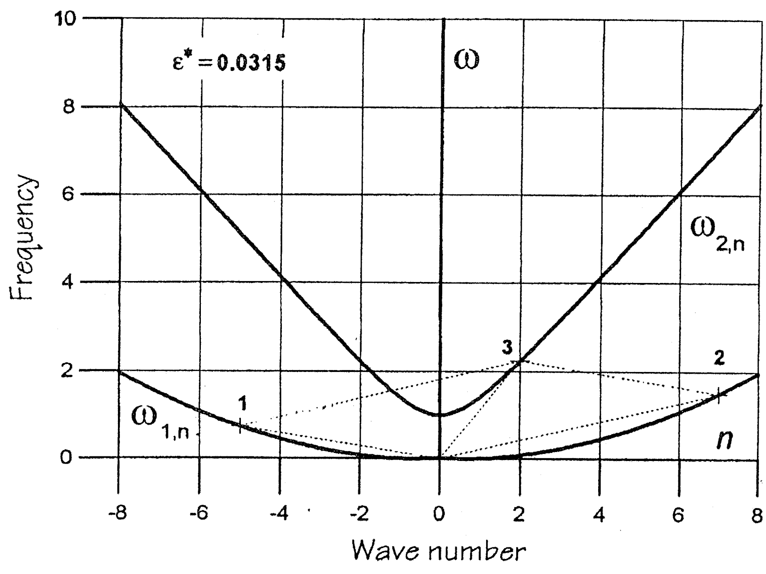


Fig. 6. Spectrum and three-wave resonance for a closed circular ring.

### 3.1. Three-wave non-linear coupling

System (3.5) is prone to exhibiting resonance couplings inside wave triads. Exact phase matching can be realized as in Fig. 6—here between a high-frequency azimuthal wave and two low-frequency bending waves traveling in the same direction. A particular case of this is given in Fig. 7 describing a so-called two-to-one internal resonance between the axisymmetric oscillation and two bending waves. A third type of resonance exists where resonance occurs between a high-frequency azimuthal wave and two low-frequency satellites, one being a bending wave and the other an azimuthal one. This last type exists only in this ring case, so that it disappears in the limit as the ring, cut properly, becomes a straight bar as its curvature goes to zero. The algebra in the ring case is somewhat similar to that performed in Section 2 and will not be repeated. Phase matching conditions read

$$\omega_1 + \omega_2 = \omega_3 + \mu\Delta\omega, \qquad n_1 \pm n_2 = n_3, \tag{3.13}$$

where $\Delta\omega$ is the detuning (equal to zero in exact matching conditions). Solutions of Eq. (3.5) are looked for in the following asymptotic form

$$\begin{aligned} v(\varphi, t) &= -\mathrm{i}\sum_{k=1}^{3} p_k A_k(\tau)\exp\mathrm{i}\Phi_k + \mu v^{(1)}(\varphi, t) + 0(\mu^2 t) + (\cdot\cdot)^*, \\ w(\varphi, t) &= \sum_{k=1}^{3} A_k(\tau)\exp\mathrm{i}\Phi_k + \mu w^{(1)}(\varphi, t) + 0(\mu^2 t) + (\cdot\cdot)^*, \end{aligned} \tag{3.14}$$

at order $\mu$ with $\tau$ the slow time scale. Substituting from these into (3.5) and equating the terms of order $\mu$, after integration over the wave phases we find the differential equations that govern the non-resonant corrections $v^{(1)}$ and $w^{(1)}$ as

$$\frac{\mathrm{d}A_k}{\mathrm{d}\tau} = -\mathrm{i}\omega_k^{-1}\Psi_k^{-2}\alpha\frac{\partial U}{\partial A_k^*}, \tag{3.15}$$

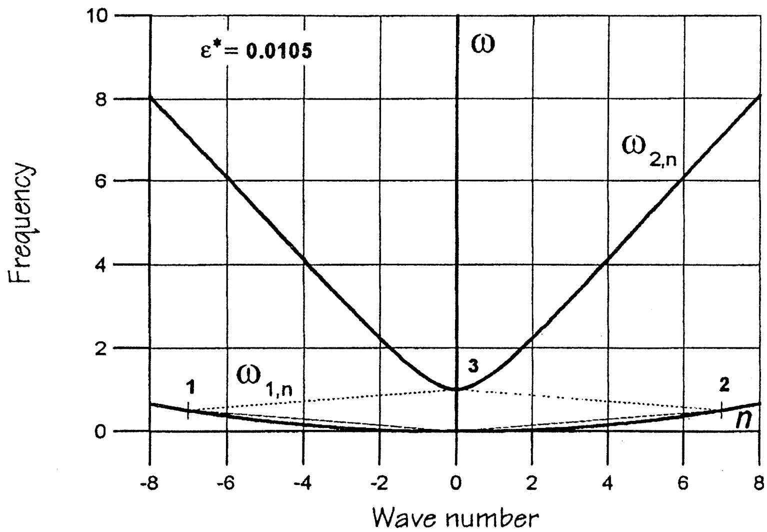


Fig. 7. Two-to-one internal resonance between axisymmetric oscillations and bending waves.

where $\alpha = W_1W_2V_3 - W_1V_2W_3$ is the non-linearity coefficient (with $V_k = 1 - p_kn_k$, $W_k = n_k - p_k$), $\Psi_k = (1 + p_k^2)^{1/2}$, and $U$ is the average potential given by

$$U = A_1A_2A_3^*\text{expi}(\Delta\omega)\tau + A_1^*A_2^*A_3\text{expi}(-\Delta\omega)\tau. \tag{3.16}$$

Initial conditions associated to (3.15) read

$$A_n(0) = a_{n0}, \quad n = 1, 2, 3. \tag{3.17}$$

Just like in Section 2, Eq. (3.15) possess first integrals in the form of the energy conservation and the Manley-Rowe relations (cf. Kovriguine et al., 2001, for details). The system being conservative (Hamiltonian), the following conclusions can be drawn in so far as stability properties—at this order of approximation—are concerned:

(i) azimuthal high-frequency waves are unstable with respect to small perturbations (so-called *break-up instability*).
(ii) bending low-frequency waves are stable with respect to small perturbations.
(iii) one can study the time evolution of the amplitude envelopes (Kovriguine et al., 2001)
(iv) The loss of stability against the high-frequency wave and the resonant excitation of two low-frequency waves is accompanied by a *stress amplification* phenomenon.

Notice that one can play with the value of the parameter $\varepsilon$—by adjusting it—so that matching conditions can be exactly satisfied. This critical value of $\varepsilon$ is found by solving the general expression (3.12) for $\varepsilon$ when we want (for instance) degenerate resonant conditions such as (this is realized in Fig. 7)

$$n = n_1 = -n_2; \; n_3 = 0, \quad \omega_1 = \omega_2 = \tfrac{1}{2}\omega_3. \tag{3.18}$$

### 3.2. Four-wave resonant coupling

Just like in Section 2 one can go further in the asymptotic analysis by including terms of order $\mu^2$ in the ansatz for the solutions (Fig. 8). Then a detuning of order $\mu^2$ is considered with general matching conditions

$$\omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 = \mu^2\Delta\omega, \qquad n_1 \pm n_2 \pm n_3 \pm n_4 = 0. \tag{3.19}$$

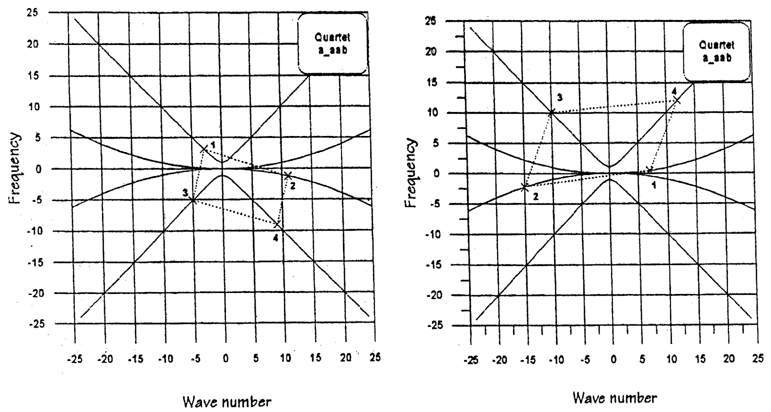


Fig. 8. Two cases of four-wave resonances in a closed circular ring.

Two such situations are shown in Fig. 8. There exist other possibilities of such *resonant quartets* (see Fig. 13 in Kovriguine et al., 2001). The study of this resonance phenomenon requires a lengthy algebra. It suffices to note that the considered system is still Hamiltonian and thus conserves energy and an energy partition between modes is characterized by appropriate Manley-Rowe relations.

### 3.3. Non-resonant interactions

These take place when the resonant interactions between three and four waves are absent. Such interactions are ubiquitous in weakly non-linear oscillatory systems. The matching conditions—for a wave couple—then look like degenerate forms of four-wave matching conditions (compare to (3.19)), e.g., trivially

$$\omega_1 - \omega_1 + \omega_2 - \omega_2 = 0, \qquad n_1 - n_1 + n_2 - n_2 = 0. \tag{3.20}$$

We refer to the original work (Kovriguine et al., 2001) for more details. Special cases of these interactions are provided by *cross-interactions of wave couples* and *self-modulation* of isolated waves. For the last case, a caricature of the matching conditions is given by (3.20) in which all $n$ and $\omega$ will bear the same index.

## 4. Conclusion

The one-dimensional structural examples briefly examined in this contribution have revealed the essential properties of three-wave and four-wave resonance couplings. Although the emphasis has been placed on mechanical consequences of these couplings, the analogy with non-linear optical systems (of which the jargon is often used) is more than obvious. This may be even more true when dealing with two-dimensional mechanical systems such as plates and shells, an example of which will be dealt with in Part 2 of this work to be published elsewhere (Kovriguine et al., 2002).

## Acknowledgements

This work was performed within the INTAS program 96-2370 and 961-2370.

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